ON SEGRE'S BOUND FOR FAT POINTS IN $\mathbb{P}^n$

EDOARDO BALLICO, OLIVIA DUMITRESCU, AND ELISA POSTINGHEL

ABSTRACT. For a scheme of fat points Z defined by the saturated ideal $\mathcal{I}_Z$, the regularity index computes the Castelnuovo-Mumford regularity of the Cohen-Macaulay ring $R/\mathcal{I}_Z$. For points in "general position" we improve the bound for the regularity index computed by Segre for $\mathbb{P}^2$ and generalised by Catalisano, Trung and Valla for $\mathbb{P}^n$. Moreover, we prove that the generalised Segre's bound conjectured by Fatabbi and Lorenzini holds for $n+3$ arbitrary points in $\mathbb{P}^n$. We propose a modification of Segre's conjecture for arbitrary points and we discuss some evidences.

1. Introduction

Let $S=\{p_1,\dots,p_s\}$ be a set of distinct points in $\mathbb{P}^n=\mathbb{P}^n_K$ and let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ be the associated homogeneous prime ideals in the polynomial ring $R:=K[x_0,\dots,x_n]$, where K is an algebraically closed field. Given positive integers $m_1,\dots,m_s$ we denote by $Z:=\sum_{i=1}^s m_ip_i$ the 0-dimensional subscheme of $\mathbb{P}^n$ defined by the saturated ideal $\mathcal{I}_Z:=\mathfrak{p}_1^{m_1}\cap\cdots\cap\mathfrak{p}_s^{m_s}$. We denote by $Z_{\mathrm{red}}:=\sum_{i=1}^s p_i$ the support of Z and by $\mathrm{w(Z)}:=\sum_{\mathrm{i=1}}^{\mathrm{s}}\mathrm{m_i}$ its weight.

Computing the value of the Hilbert function of $\mathcal{I}_Z$ at d is equivalent to computing the dimension of the linear systems $\mathcal{L}_{n,d}(m_1,\dots,m_s)$ of the degree-d hypersurfaces of $\mathbb{P}^n$ passing through each point p_i with multiplicity at least m_i, for all $d\geq 0$.

The *regularity index* $\mathrm{reg(Z)}$ of Z is the smallest positive integer d such that $h^1(\mathbb{P}^n,\mathcal{I}_Z(d))=0$ or, equivalently, $h^1(\mathbb{P}^n,\mathcal{L}_{n,d}(m_1,\dots,m_s))=0$. This number corresponds to the *Castelnuovo-Mumford regularity* of the Cohen-Macaulay graded ring $R/\mathcal{I}_Z$.

1.1. **Segre's bound.** Let us assume, without loss of generality, that $m_1\geq\cdots\geq m_s\geq 1$.

In 1961, Segre [20] gave the following upper bound for the regularity index of a collection Z of fat points in general position in $\mathbb{P}^2$:

$$\mathrm{reg(Z)}\leq\max\left\{\mathrm{m_1+m_2-1},\left\lfloor\frac{\mathrm{w(Z)}}{2}\right\rfloor\right\}. \tag{1}$$

We must also mention that for plane points in general position, namely such that not three of them lie on a line, not six of them lie on a conic etc., that was Segre's original hypothesis, the bound for the regularity index corresponds to the the famous conjecture of Segre, Harbourne, Gimigliano and Hirschowitz for linear systems of plane curves with fixed multiple base points.

2010 *Mathematics Subject Classification.* Primary: 13D40 Secondary: 14C20.

Key words and phrases. Segre's bound, generalised Segre's bound, regularity index, fat points.

The first author was partially supported by MIUR and GNSAGA of INdAM (Italy). The second author is member of "Simion Stoilow" Institute of Mathematics of the Romanian Academy. The third author is supported by the Research Foundation - Flanders (FWO).

In 1991, Catalisano [5, 6] established that the bound (1) holds sharp for sets of points that are three by three not collinear. See [15, 16, 23] for discussions about Segre's bound for fat points satisfying stronger conditions.

For arbitrary fat points in $\mathbb{P}^2$, in 1969 Fulton [12] gave the following upper bound:

$$\mathrm{reg(Z) \leq w(Z) - 1.} \tag{2}$$

It was proved to be sharp if and only if all points lie on a line by Davis and Geramita [10] in 1984.

Fatabbi in 1994 [11] proved that reg(Z) is bounded above by the maximum between the number $\lfloor w(Z)/2 \rfloor$ and the maximal sum of the multiplicities of collinear points of S.

The above results were extended to fat points in *linearly general position* in $\mathbb{P}^n$. Fix $n \geq 2$, $s \geq 2$. We say the points $p_1, \ldots, p_s$ are in linearly general position in $\mathbb{P}^n$ if for each integer $r \in \{1, \ldots, n-1\}$ we have $\sharp(S \cap L) \leq r+1$, for all r-dimensional linear subspaces $L \subset \mathbb{P}^n$.

Catalisano, Trung and Valla in [9, Theorem 6] showed that if Z is a collection of fat points in linearly general position in $\mathbb{P}^n$, then

$$\mathrm{reg(Z) \leq \max\left\{m_1 + m_2 - 1, \left\lfloor \frac{w(Z) + n - 2}{n} \right\rfloor\right\}.} \tag{3}$$

Moreover they proved that the bound is sharp for $s \leq n+2$ points in linearly general position and, for $s \geq n+3$, when the points lie on a rational normal curve ([9, Proposition 7]). See also [7, 8].

The bound (3) is nowadays referred to as *Segre's bound* for the regularity index of a collection of fat points Z in $\mathbb{P}^n$.

1.2. **Generalised Segre's bound.** For arbitrary fat points in $\mathbb{P}^n$, Fatabbi and Lorenzini [13] gave the following conjecture for the regularity index.

For any subset $L \subseteq \mathbb{P}^n$, write $\mathrm{w_L(Z)}$ for the sum of all m_p, where $p \in S \cap L$ and m_p is the multiplicity of Z at p. In particular $\mathrm{w}_{\mathbb{P}^n}\mathrm{(Z)}$ is the weight of Z, w(Z).

Conjecture 1.1. *For $r = 1, \ldots, n$ and for any linear r-subspace L of $\mathbb{P}^n$, set*

$$T(Z, L) := \left\lfloor \frac{\mathrm{w_L(Z) + r - 2}}{r} \right\rfloor.$$

Then

$$\mathrm{reg(Z) \leq \max\{T(Z, L) : L \subseteq \mathbb{P}^n\}.} \tag{4}$$

The bound in (4) is referred to as *generalised Segre's bound* for the regularity index of an arbitrary collection of fat points in $\mathbb{P}^n$. Notice that for schemes of fat points in linearly general position, the generalised Segre's bound (4) equals precisely Segre's bound (3). The case $n = 2$ was proved to be true in [11].

Conjecture 1.1 was established in the case $n = 3$ by Thiên in 2000 [22] and, independently, by Fatabbi and Lorenzini in 2001 [13]. The first author also proved the case of arbitrary double points in $\mathbb{P}^4$.

More recently, Benedetti, Fatabbi and Lorenzini in [2] proved that the conjecture holds for arbitrary $s \leq n+2$ points of $\mathbb{P}^n$.

Successively, Tu and Hung [24] showed that Conjecture 1.1 holds for $n+3$ points that are *almost equimultiple*, namely when $m_i \in \{m-1, m\}$, for all $i = 1, \ldots, n+3$.

In Section 2 we prove that Conjecture 1.1 holds for schemes with $n+3$ arbitrary fat points of $\mathbb{P}^n$.

In Section 3, Theorem 3.6, that is based on the results of Brambilla, Dumitrescu and Postinghel [3], improves *Segre's bound* (3) (and also (4)) for fat points in general position in $\mathbb{P}^n$. An instance of this is the scheme of seven double points in $\mathbb{P}^3$. In this case, Segre's bound (3) is 5, but $\mathcal{L}_{3,4}(2^7)$ has vanishing first cohomology group, as predicted by the bound (11) given in Theorem 3.6, that is 4. This is a well-known example that follows from the Alexander-Hirschowitz theorem [1].

In Section 4 we pose a modification of the Segre conjecture for the regularity index of a scheme of fat points, $\mathrm{reg(Z)}$, and prove it holds for $n = 3$.

Acknowledgements. The second and third authors would like to thank the Research Center FBK-CIRM Trento for the hospitality and financial support during their one month "Research in Pairs", Winter 2015. This project was initiated during this program. We are grateful to the referee for his/her useful comments.

2. Generalised Segre's bound for $n + 3$ arbitrary points

In this section we prove that Conjecture 1.1 is true for an arbitrary collection Z of $n+3$ fat points in $\mathbb{P}^n$. We recall that a *non-degenerate* set of points in $\mathbb{P}^n$ is one whose linear span is the whole space $\mathbb{P}^n$.

Theorem 2.1. *Let $Z := \sum_{i=1}^{n+3} m_i p_i$ be a scheme of fat points supported on a non-degenerate set of distinct points in $\mathbb{P}^n$. Then Z satisfies the generalised Segre's bound, namely*

$$\mathrm{reg(Z)} \le \max\{\mathrm{T(Z,L)} : \mathrm{L} \subseteq \mathbb{P}^{\mathrm{n}}\}.$$

In order to prove this result, we need the following lemmas.

Lemma 2.2. *Fix an integer $a \ge 3$, hyperplanes $H, M \subset \mathbb{P}^4$, $H \ne M$, and sets $S_1 \subset H \cap M$, $S_2 \subset H \setminus (H \cap M)$, $S_3 \subset M \setminus (H \cap M)$ such that $\sharp(S_1) = 3$, $\sharp(S_2) = \sharp(S_3) = 2$, $S_1 \cup S_2$ is in linearly general position in H and $S_1 \cup S_3$ is in linearly general position in M. Set $S := S_1 \cup S_2 \cup S_3$ and $Z_a := \sum_{p \in S} ap$. Then $h^1(\mathcal{I}_{Z_a}(2a-1)) = 0$.*

Proof. Note that, up to a projective transformations, S is uniquely determined. The proof will be by induction on a.

The case $a = 3$ is an explicit computation, that can be easily performed with the help of a computer. If $e_0, \dots, e_4$ are the coordinate points of $\mathbb{P}^4$, one can choose H to be the hyperplane spanned by the points $\{e_0, e_1, e_2, e_3\}$, M to be the hyperplane spanned by the points $\{e_0, e_1, e_2, e_4\}$, $S_1 = \{e_0, e_1, e_2\}$, $S_2 = \{e_3, e_0 + e_1 + e_2 + e_3\}$ and $S_3 = \{e_4, e_0 + e_1 - e_2 + e_4\}$. In this example it is easy to check that $h^1(I_{Z_3}(5)) = 0$ (see Section A.1).

Now assume $a > 3$. We have $h^1(\mathcal{I}_S(2)) = 0$. One can easily check this by studying the Castelnuovo residual sequence with respect to H. Now we check that $\mathcal{I}_S(2)$ is spanned. Using a residual exact sequence with the quadric hypersurface $H \cup M$ we get $h^1(\mathcal{I}_{S \cup \{o\}}(2)) = 0$ for all $o \notin H \cup M$, i.e. $\mathcal{I}_S(2)$ is spanned outside $H \cup M$. Then using a residual exact sequence with respect to H (resp. M) we see that $\mathcal{I}_S(2)$ is spanned at each point of $H \setminus (S \cap H)$ (resp. $M \setminus (S \cap M)$). Now fix $o \in S$, say $o \in S \cap H$. Using the residual exact sequence of H we get $h^1(\mathcal{I}_{S \setminus \{o\} \cup 2o}(2)) = 0$ and hence $\mathcal{I}_S(2)$ is globally generated at o. Since $\mathcal{I}_S(2)$ is spanned and S is finite, Bertini's theorem gives the existence of smooth quadric hypersurfaces $Q, Q, Q' \in |\mathcal{I}_S(2)|$ such that $Q \cap Q'$ is a smooth surface and $C := Q \cap Q' \cap Q''$ is a smooth curve. By Lefschetz' Theorem, C is irreducible. By

the adjunction formula C is a canonically embedded smooth curve of genus 5. The inductive assumption gives $h^1(\mathcal{I}_{Z_{a-1}}(2a-3)) = 0$ and so $h^1(Q, \mathcal{I}_{Z_{a-1}\cap Q}(2a-3)) = 0$ and $h^1(Q, \mathcal{I}_{Z_{a-1}\cap Q\cap Q'}(2a-3)) = 0$. The residual exact sequence

$$0 \to \mathcal{I}_{Z_{a-1}}(2a-3) \to \mathcal{I}_{Z_a}(2a-1) \to \mathcal{I}_{Z_a\cap Q}(2a-1) \to 0$$

shows that it is sufficient to prove that $h^1(Q, \mathcal{I}_{Z_a\cap Q}(2a-1)) = 0$. The residual exact sequence

$$0 \to \mathcal{I}_{Z_{a-1}\cap Q,Q}(2a-3) \to \mathcal{I}_{Z_a\cap Q}(2a-1) \to \mathcal{I}_{Z_a\cap Q\cap Q'}(2a-1) \to 0$$

shows that it is sufficient to prove that $h^1(Q\cap Q', \mathcal{I}_{Z_a\cap Q\cap Q',Q\cap Q'}(2a-1)) = 0$. The residual exact sequence

$$0 \to \mathcal{I}_{Z_{a-1}\cap Q\cap Q',Q\cap Q'}(2a-3) \to \mathcal{I}_{Z_a\cap Q\cap Q'}(2a-1) \to \mathcal{I}_{Z_a\cap C,C}(2a-1) \to 0$$

shows that it is sufficient to prove that $h^1(C, \mathcal{I}_{Z_a\cap C,C}(2a-1)) = 0$. Since C is a complete intersection, it is projectively normal. Thus it is sufficient to prove that $h^1(C, R) = 0$, where R is the line bundle $\mathcal{O}_C(2a-1)(-Z_a\cap C)$. We have $h^1(C, R) = 0$, because C has genus 5 and the Euler characteristic of R is $\chi(R) = 8(2a-1) - 7a \geq 9$. □

Remark 2.3. Lemma 2.2 is false for $a = 2$, even if we take S to be a set of general points in $\mathbb{P}^4$ (it is an exceptional case in the list of Alexander-Hirschowitz [1]. See also [4, 19]). Also note that for $a = 2$ the Segre number is 4, because $\lfloor \frac{7\cdot 2+2}{4} \rfloor = 4$.

Lemma 2.4. *Fix integers $n \geq 2$, $m > 0$, $a > 0$ and $t \geq a+m-1$. Fix a finite set $S \subset \mathbb{P}^n$ and $o \in S$. Fix a hyperplane $H \subset \mathbb{P}^n$ such that $o \notin H$. Let $\ell : \mathbb{P}^n \setminus \{o\} \to \mathbb{P}^{n-1}$ be the linear projection from o. Set $S' := S \setminus \{o\}$ and $S_1 := \ell(S')$. Assume that $\ell|S'$ is injective. For all $p \in S$ fix an integer $m_p \geq 0$ with the restriction that $m_o = m$ and that $m_p \leq a$ for each $p \in S'$. Set $Z := \sum_{p\in S} m_p p$. For each integer $x \geq 0$ set $W_x := \sum_{p\in S'} \max\{0, m_p - x\}\ell(p)$ (hence $W_x = \emptyset$ for all $x \geq a$). Assume $h^1(H, \mathcal{I}_{W_x}(t-x)) = 0$ for all $x = 0, \dots, a$. Then $h^1(\mathcal{I}_Z(t)) = 0$.*

Proof. For each integer $x \geq 0$ set $Z_x := mo + \sum_{p\in S'} \max\{m_p - x\}\ell(p)$. We have $W_x = Z_x \cap H$ and $Z_x = mo$ for all $x > 0$. Choose a system $x_0, \dots, x_n$ of homogeneous coordinates such that $H = \{x_0 = 0\}$ and $o = (1 : 0 : \cdots : 0)$. For each $\lambda \in \mathbb{K} \setminus \{0\}$ let $h_\lambda : \mathbb{P}^n \to \mathbb{P}^n$ be the automorphism defined by the formula $h_\lambda(x_0 : x_1 : \cdots : x_n) = (\lambda x_0 : x_1 : \cdots : x_n)$. We have $h_\lambda(o) = o$. Since h_λ is an automorphism of $\mathbb{P}^n$, Z and $h_\lambda(Z)$ have the same Hilbert function. Since Z_0 is a flat limit of the family $\{h_\lambda(Z)\}_{\lambda\neq 0}$, it is sufficient to prove that $h^1(\mathcal{I}_{Z_0}(t)) = 0$. For each integer $x = 0, \dots, a$ there is a residual exact sequence

$$0 \to \mathcal{I}_{Z_{x+1}}(t-x-1) \to \mathcal{I}_{Z_x}(t-x) \to \mathcal{I}_{W_x,H}(t-x) \to 0.$$

We can conclude using the assumptions on W_x and that $h^1(\mathcal{I}_{Z_a}(t-a)) = h^1(\mathcal{I}_{mo}(t-a)) = 0$, because $t \geq a+m-1$. □

Lemma 2.5. *Fix integer $t \geq 2$, $z > z_1 \geq \cdots \geq z_t > 0$ and set $\eta := z + z_1 + 1$. Then $z + z_1 + \cdots + z_t \leq \frac{t+1}{2}\eta - t$ if η is even and $z + z_1 + \cdots + z_t \leq \frac{t+1}{2}\eta - (t-1)/2$ if η is odd.*

Proof. It is sufficient to prove the statement when $z_i = z_1$ for all i. We fix η. If η is even, the left hand side of the inequality is maximal if $z = \eta/2$ and $z_1 = \eta/2 - 1$ and in this case we have $z + tz_1 = (t+1)\eta/2 - t$. If η is odd, then the left hand

side of the inequality is maximal if $z = (\eta+1)/2$ and $z_1 = (\eta-1)/2$ and in this case we have $z + tz_1 = (t+1)(\eta+1)/2 - t$. □

Proof of Theorem 2.1. We will use the notation $S := Z_{\mathrm{red}}$. We will denote by α the Segre bound for reg(Z).

The proof is by induction on n and w(Z). The case $n = 1$ is obvious by the cohomology of line bundles on $\mathbb{P}^1$. Hence we may assume $n \geq 2$. Now assume that w(Z) is as minimal as possible, i.e. $\mathrm{w(Z) = n+3}$, i.e. $m_i = 1$ for all i. Since S spans $\mathbb{P}^n$, we need to prove that $h^1(\mathcal{I}_S(3)) = 0$ if there is a line $L \subset \mathbb{P}^n$ with $\sharp(S \cap L) = 4$ and $h^1(\mathcal{I}_S(2)) = 0$ if there is no such a line. Let $H \subset \mathbb{P}^n$ be a hyperplane such that $\sharp(S \cap H)$ is maximal. In particular $S \cap H$ spans H. Since S spans H, we have $n+1 \leq \sharp(S \cap H) \leq n+2$. Therefore $h^1(\mathcal{I}_{S \setminus S \cap H}(1)) = 0$. If $L \subset \mathbb{P}^n$ is a line with $\sharp(S \cap L) \geq 4$, then $L \subset H$. The inductive assumption gives that $h^1(H, \mathcal{I}_{S\cap H}(3)) = 0$ and that $h^1(H, \mathcal{I}_{S\cap H}(2)) = 0$ if there is no line L with $\sharp(S \cap L) = 4$.

Hence we may use induction on w(Z).

If S is in linearly general position, the statement is a particular case of [9, Theorem 6]. From now on we will assume that the points are not in linearly general position. In particular we will handle separately the two following cases.

Case (1): $n+2$ points of S are contained in a hyperplane.

Case (2): $n+1$ points of S are contained in a hyperplane, but no hyperplane contains $n+2$ points.

Case (1).

After relabelling the points if necessary, we may assume that $H \cong \mathbb{P}^{n-1}$ is a hyperplane such that $p_1, \ldots, p_{n+2} \in H$ and $p_{n+3} \notin H$. Set $W := \mathrm{Res}_H(Z) = \sum_{i=1}^{n+2}(m_i - 1)p_1 + m_{n+3}p_{n+3}$. Let β be the Segre bound for reg(W). Consider the residual exact sequence

$$0 \to \mathcal{I}_W(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z\cap H, H}(d) \to 0. \tag{5}$$

It is sufficient to prove that $\mathrm{reg(W)} \leq \alpha - 1$ and $\mathrm{reg(Z \cap H)} \leq \alpha$. The latter is obvious, because by the inductive assumption on n and the fact that the linear subspaces arising in the test of the bound for $Z \cap H$ are some of the ones used in the definition of α. By the inductive assumption on w(Z), we may assume that W satisfies the statement, namely that $\mathrm{reg(W)} \leq \beta$. Therefore it suffices to prove that $\beta \leq \alpha - 1$. For any linear space $M \subseteq \mathbb{P}^n$ we have $\mathrm{w_M(W) = w_M(Z) - \sharp(S \cap M \cap H)}$.

Let $L \subseteq \mathbb{P}^n$ be a linear r-subspace evincing β, i.e. $\beta = T(W, L)$. It is a subspace spanned by the points of W_{red} and in particular by points of S. Since $L \subset \mathbb{P}^n$, the definitions of α and of $\mathrm{w_L}$ give $\mathrm{w_L(Z) \leq r\alpha + 1}$. Since $\sharp(S \cap H) = \sharp(S) - 1$, we get $\mathrm{w_L(W) \leq w_L(Z) - r \leq r\alpha + 1 - r}$ and hence $\beta \leq \alpha - 1$.

Case (2). We may assume that H is a hyperplane of $\mathbb{P}^n$ with $p_1 \ldots, p_{n+1} \in H$ and $p_{n+2}, p_{n+3} \notin H$. Similarly to Case (1), set $W := \mathrm{Res}_H(Z) = \sum_{i=1}^{n+1}(m_i - 1)p_i + m_{n+2}p_{n+2} + m_{n+3}p_{n+3}$. By the inductive assumptions and the residual exact sequence (5) it suffices to prove that $\beta \leq \alpha - 1$.

Let $L \subseteq \mathbb{P}^n$ be a linear r-subspace evincing β. It is spanned by points in the support of W and in particular it is spanned by points of S. We have $\beta = \lfloor \frac{\mathrm{w_L(W)+r-2}}{r} \rfloor$ and $\mathrm{w_L(W) = w_L(Z) - \sharp(S \cap L \cap H)}$. Since $\mathrm{w_L(Z) \leq r\alpha + 1}$, we have $\beta \leq \alpha - 1$, unless $\sharp(S \cap L \cap H) \leq r - 1$, i.e. unless L contains p_{n+2} and p_{n+3}. Assume that this is the case. We have $\sharp(S \cap L \cap H) = r - 1$.

We consider the following cases.

Case (a) L has dimension $r \geq 2$.
Case (b) L has dimension 1.

Case (a).

In this case $\dim(L) \geq 2$, we claim that $\mathrm{w_L(W)} \leq 2(\mathrm{r}+1)$. If $\mathrm{w_L(W)} = 2(\mathrm{r}+1)$ then $\beta = 3$ and, moreover, $m_{n+2} + m_{n+3} \leq 4$, as $m_{n+2} + m_{n+3} - 1 \leq \beta$, by the definition of β. Hence we can conclude that $\alpha \geq \beta + 1 = 4$. Indeed, let $L' \subset L \cap H$ be the linear space spanned by $S \cap L \setminus \{p_{n+2}, p_{n+3}\}$; notice that $\dim(L') = r - 2$. Then we have $\alpha \geq T(Z, L') = \lfloor (\mathrm{w_{L'}(Z)}+\mathrm{r}-4)/(\mathrm{r}-2) \rfloor = \lfloor (\mathrm{w_L(W)}+(\mathrm{r}-1)-\mathrm{m_{n+2}}-\mathrm{m_{n+3}}+\mathrm{r}-4)/(\mathrm{r}-2) \rfloor \geq \lfloor (\mathrm{w_L(W)}+(\mathrm{r}-1)-4+\mathrm{r}-4)/(\mathrm{r}-2) \rfloor = 4$. If $\mathrm{w_L(W)} < 2(\mathrm{r}+1)$, then $\beta < 3$. In this case we have $m_{n+2} + m_{n+3} \leq 3$, hence $\max\{m_{n+2}, m_{n+3}\} \leq 2$. Moreover, since $r \geq 2$, it must be $m'_i \geq 1$ for some $p_i \in S \cap L \cap H$, $i \neq n+2, n+3$. Therefore $m_i \geq 2$ and this implies that $\alpha \geq m_i + \max\{m_{n+2}, m_{n+3}\} - 1 \geq 3 \geq \beta + 1$ and we conclude.

We are left with proving the claim. Let I be the index set parametrizing the union of points $S \cap L$. Set $m'_i = m_i - 1$, for all $i \in I \setminus \{n+2, n+3\}$, and $m'_{n+2} = m_{n+2}$, $m'_{n+3} = m_{n+3}$, so that $\mathrm{w_L(W)} = \sum_{\mathrm{i} \in \mathrm{I}} \mathrm{m'_i}$. The definition of β implies that

$$\frac{\mathrm{w_L(W)} + \mathrm{r} - 2}{r} \geq m'_i + m'_j - 1,$$

for all $i, j \in I$, $i \neq j$. If $r = 2\rho - 1$ ($\rho \geq 2$), then

$$\begin{aligned} \rho \frac{\mathrm{w_L(W)} + \mathrm{r} - 2}{r} &\geq (m'_1 + m'_2 - 1) + \cdots + (m'_{n+2} + m'_{n+3} - 1) \\ &= \mathrm{w_L(W)} - \rho. \end{aligned}$$

One can easily check that this is equivalent to $\mathrm{w_L(W)} \leq 2(\mathrm{r}+1)$. If instead $r = 2\rho$ ($\rho \geq 1$), by a similar computation one obtains

$$\rho \frac{\mathrm{w_L(W)} + \mathrm{r} - 2}{r} \geq \mathrm{w_L(W)} - \rho - \mathrm{m'_i},$$

for all $i \in I$. We leave it to the reader to check that by taking the sum over $i \in I$ of the above expressions, one concludes that $\mathrm{w_L(W)} \leq 2(\mathrm{r}+1)$ also in this case.

Case (b).

Assume that L is the line spanned by p_{n+2}, p_{n+3} and that $\sharp(S \cap L) = 2$. If α is not attained by the line L, then $\alpha > m_{n+2} + m_{n+3} - 1 = \beta$ and we conclude. Assume now that α is attained by the line spanned by p_{n+2}, p_{n+3}, i.e. $\alpha = m_{n+2} + m_{n+3} - 1$.

Without loosing generality we may assume $m_{n+3} \geq m_{n+2}$. The definition of α gives $m_i \leq m_{n+2}$ for all $i \leq n+1$. Hence, up to a permutation of the first $n+1$ indices we may assume that the sequence m_i, $1 \leq i \leq n+3$, is non-increasing. If $m_{n+3} > m_{n+2}$, then the lines R with $\mathrm{w_R(Z)} = \alpha + 1$ are spanned by p_{n+3} and the points p_i with $m_i = m_{n+2}$. In this case all such lines contain p_{n+3}. If $m_{n+3} = m_{n+2}$ (and hence α is odd), then the lines R with $\mathrm{w_R(Z)} = \alpha + 1$ are the lines spanned by two points with multiplicity $m_{n+3} = (\alpha+1)/2$.

We will split the proof of the statement in the following cases.

Case (b.1): homogeneous case, i.e. $m_1 = ... = m_{n+3} = m$.

Case (b.2): there exists a subspace N of $\dim(N) \leq n-2$ containing $\dim(N)+2$ points of S.

Case (b.3): every subspace N, with $\dim(N) \leq n-2$, contains $\dim(N)+1$ points of S.

Case (b.1).

If $m = 1$, we have $\alpha > 1$. Since S spans $\mathbb{P}^n$, we have $\sharp(S \cap N) \leq \dim(N) + 3$ for all linear spaces $N \subsetneq \mathbb{P}^n$. Since $n \geq 4$ we see that $\alpha = 3$ if $\sharp(S \cap R) = 4$ for some line R and $\alpha = 2$ in all other cases. Notice that the first case does not occur. Finally, if $\alpha = 2$ the vanishing of $h^1(\mathcal{I}_S(\alpha))$ is well-known to hold.

Assume $m \geq 2$, for all $i = 1, \dots, n+3$, and $\alpha = 2m - 1$. For each integer $t = 1, \dots, n-1$ let γ_t be the maximal integer such that $\gamma_t(\alpha+1)/2 \leq t\alpha + 1$. We have $\gamma_t = t+1$ for $t = 1, 2$, $\gamma_3 = 5$ and $\gamma_t \geq t+3$ for all $t \geq 4$. Since α is the Segre bound of Z, we have $\sharp(S \cap N) \leq \gamma_t$ for each t-dimensional linear space $N \subset \mathbb{P}^n$. In particular $\sharp(S \cap N) \leq 2$ for each line N and $\sharp(S \cap N) \leq 3$ for each plane N. By assumption, $\sharp(S \cap N) \leq \dim(N) + 2$ for each linear space $N \subsetneq \mathbb{P}^n$. Fix any hyperplane M such that $\sharp(S \cap M) = n + 1$ and set $S' := S \cap M$. Call q, q' the points of $S \setminus (S \cap M)$ and D the line spanned by q and q'. Set $\{o\} := D \cap M$ and $S'' := S' \cup \{o\}$. Since $\sharp(S \cap N) \leq 2$ for each line N, we have $o \notin S'$. Set

$$Z' := \frac{\alpha+1}{2}o + \frac{\alpha+1}{2}q + \sum_{P \in S'} \frac{\alpha+1}{2}q.$$

Choose a system $x_0, \dots, x_n$ of homogeneous coordinates such that $M = \{x_0 = 0\}$ and $q = (1 : 0 : \cdots : 0)$. For each $\lambda \in \mathbb{K} \setminus \{0\}$ let $h_\lambda : \mathbb{P}^n \to \mathbb{P}^n$ be the automorphism defined by the formula $h_\lambda(x_0 : x_1 : \cdots : x_n) = (\lambda x_0 : x_1 : \cdots : x_n)$. We have $h_\lambda(q) = q$ and $h_\lambda(p) = p$ for each $p \in M$. Since h_λ is an automorphism, Z and $h_\lambda(Z)$ have the same regularity. Since Z' is a flat limit of the family $\{h_\lambda(Z)\}_{\lambda \neq 0}$, the semicontinuity theorem for cohomology gives $\mathrm{reg}(Z') \geq \mathrm{reg}(Z)$. Hence is sufficient to prove that $h^1(\mathcal{I}_{Z'}(\alpha)) = 0$. Therefore we are done if α is Segre's bound for Z'. By the shape of the function γ_t or by the proof of Case 2 we see that it is sufficient to prove that $Z' \cap M$ has index of regularity α. By the inductive assumption on n and the shape of the function γ_t we see that it is sufficient to prove that $\sharp(S'' \cap R) \leq 2$ for each line $R \subset M$ and $\sharp(S'' \cap N) \leq 3$ for each plane $N \subset M$. Assume the existence of a line $R \subset M$ such that $\sharp(S'' \cap N) \geq 3$. Since $\sharp(S' \cap R) \leq \gamma_1 = 2$, we have $o \in R$ and $\sharp(S' \cap R) = 2$. Hence $R \cup D$ spans a plane A with $\sharp(S \cap A) = 4 > \gamma_2$, contradicting the assumption that α is the Segre bound of Z.

Now assume the existence of a plane N such that $\sharp(S'' \cap N) \geq 4$. Since $\gamma_2 = 3$ we see that $o \in N$ and $\sharp(S' \cap N) = 3$. Therefore $R \cup N$ spans a 3-dimensional linear subspace A with $\sharp(S \cap A) = 5$. We repeat the construction taking a hyperplane $M(1)$ containing A and spanned by points of S. Set $S'(1) := S \cap M(1)$, $\{q(1), q(1)'\} := S \setminus S'(1)$. Call $D(1)$ the line spanned by $\{q(1), q(1)'\}$ and set $o(1) := D(1) \cap M(1)$. We repeat the construction with $M(1)$ instead of M and get $h^1(\mathcal{I}_Z(\alpha)) = 0$, unless there is a 3-dimensional linear space $A(1) \subset \mathbb{P}^n$ with $\sharp(S \cap A(1)) = 5$ and a plane $N(1) \subset M(1)$ such that the linear span of $N(1)$ and $D(1)$ is $A(1)$ and $\sharp(S \cap N(1)) = 3$. Set $b := \sharp(S \cap A \cap A(1))$. By construction we have $b \leq 3$. Let u be the the dimension of the linear span E of $A \cup A(1)$. We have $\sharp(S \cap E) \geq 10 - b$ and $u \leq 7 - b$. We get $E = \mathbb{P}^n$ and hence $n \leq 7$. We also get that $S = S \cap (A \cup A(1))$, $S \cap A \cap A(1)$ is linearly independent (it may be empty) and that $A \cap A(1)$ is spanned by $S \cap A \cap A(1)$. For each n, any two sets $S', S'(1)$ with the properties just described are projectively equivalent.

Case (b.1.1).

Assume $n = 4$. If $\alpha = 3$, then this case is excluded, because $n = 4$, $s = 7$, and $m_i = 2$ for all i has Segre number 4. If $\alpha > 3$, then we use the case $m_i = (\alpha+1)/2$ of Lemma 2.2.

Case (b.1.2).

Assume $n > 4$. Consider first the case $n = 5$, $m = 2$. We want to prove that $h^1(\mathcal{I}_Z(3)) = 0$ for any union Z of eight double points of $\mathbb{P}^5$ such that $\sharp(S \cap N) \leq \gamma_t$ for any t-dimensional subspace, with $\gamma_1 = 2$, $\gamma_2 = 3$, $\gamma_3 = 5$ and $\gamma_4 = 6$, the last inequality being sharp for a hyperplane H. It is enough to exhibit an example for which for every t there is a subspace N with $\sharp(S \cap N) = \gamma_t$ that satisfies the claim. One can show by computer that the statement holds for the set S given by $e_0, \dots, e_5$, the coordinate points, and $e_0 + e_1 + e_2 + e_3$, $e_0 + e_1 + e_5 + e_6$ (see Section A.2).

Now assume $(n, m) \neq (5, 2)$. With this restriction we know the vanishing for the pair $(n-1, m)$ by the inductive assumption on n. Fix $o \in S \cap M$ and take a general hyperplane $N \subset \mathbb{P}^n$. Let $\ell : \mathbb{P}^n \setminus \{o\} \to N$ the linear projection from o. Set $S' := S \setminus \{o\}$ and $S_1 := \ell(S')$. For all integers $x \geq 0$ set

$$W_x := \left(\sum_{p \in S_1} \max\left\{ \frac{\alpha+1}{2} - x, 0 \right\} p \right) \cap N.$$

Since S_1 is the configuration of $n + 2$ points of N corresponding to α, we have $h^1(N, \mathcal{I}_{W_0}(\alpha)) = 0$. Using Segre's bound in $N = \mathbb{P}^{n-1}$ we also get the other vanishing needed in order to apply Lemma 2.4 with $m = (\alpha + 1)/2$ and $t = \alpha$.

Case (b.2).

Let $N \subset \mathbb{P}^n$ be a minimal subspace containing exactly $\dim(N) + 2$ points of S. In this step we assume $y := \dim(N) \leq n - 2$. In this case we consider the residual sequence with respect to a hyperplane H' containing N, spanned by points of S and containing p_{n+3}. We get $h^1(\mathcal{I}_Z(\alpha)) = 0$, unless the two points, say o_1 and o_2, of $S \setminus (S \cap H')$ span a line R with $S \cap R \cap H' = \emptyset$ and $m_{o_1} + m_{o_2} = \alpha + 1$. This can not occur if $m_{n+3} > m_{n+2}$. Indeed in this case each line R with $T(R, Z) = \alpha + 1$ contains p_{n+3}. Therefore we may choose H' to be a hyperplane containing $N \cup \{p_{n+3}\}$ and obtain $p_{n+3} \in S \cap R \cap H'$ for every line R.

Assume $m_{n+3} = m_{n+2} = \frac{\alpha+1}{2}$. The scheme $\widetilde{Z} := \sum_{p \in S} \frac{\alpha+1}{2} p$ satisfies the vanishing $h^1(\mathcal{I}_{\widetilde{Z}}(\alpha)) = 0$ by Case (b1). We conclude by noticing that $Z \subset \widetilde{Z}$ implies $\mathrm{reg}(\mathrm{Z}) \leq \mathrm{reg}(\widetilde{\mathrm{Z}})$.

Case (b.3).

In this case any n points of S are linearly independent. If $m_{n+2} = m_{n+3} = \frac{\alpha+1}{2}$, we set $\widetilde{Z} := \sum_{p \in S} \frac{\alpha+1}{2} p$ and conclude by Case (b1) as above.

Assume $m_{n+3} > m_{n+2}$. Recall that since $m_{n+3} + m_{n+2} = \alpha + 1$ and no triplet of points of S is supported on a line, each line R with $\mathrm{w_R(Z)} = \alpha + 1$ is spanned by p_{n+3} and a point with multiplicity m_{n+2}. Let H' be the hyperplane spanned by $p_{n+4}, \dots, p_4$. Set $W' := \mathrm{Res}_{H'}(Z)$. By the inductive assumption it is sufficient to prove that Segre's bound for W' is at most $\alpha - 1$, i.e. that $\mathrm{w_A(W')} \leq \mathrm{t}(\alpha - 1) + 1$ for all integer $t = 1, \dots, n$ and all t-dimensional linear subspaces of $A \subseteq \mathbb{P}^n$. It is sufficient to test the linear subspaces A spanned by $S \cap A$.

If $t = n$, we prove the statement by noticing that $\mathrm{w(W')} = \mathrm{w(Z)} - \sharp(\mathrm{S} \cap \mathrm{H'}) \leq \mathrm{n}(\alpha - 1) + 1$.

Now assume $t \leq n-2$. By assumption $\sharp(S \cap A) = t + 1$. We have $\sharp(S \cap A \cap H') = t + 1 - \sharp(\{p_1, p_2, p_3\} \cap A)$ and hence $\sharp(S \cap A \cap H') < t$ only if at least two among p_1, p_2, p_3 are contained in A. First assume $\sharp(S \cap A \cap H') = t - 1$. In this case we

have $\mathrm{w_A(Z)} \le \mathrm{m_{n+3}} + \cdots + \mathrm{m_{n+5-t}} + \mathrm{m_2} + \mathrm{m_1}$ and $\mathrm{w_A(W')} \le \mathrm{m_{n+3}} + \cdots + \mathrm{m_{n+5-t}} + \mathrm{m_2} + \mathrm{m_1} - (\mathrm{t}-1)$. Lemma 2.5 with $\eta = \alpha$ gives $\mathrm{w_A(Z)} \le \frac{\mathrm{t}+1}{2}\alpha - \frac{\mathrm{t}-1}{2}$ and hence $\mathrm{w_A(W')} \le \frac{\mathrm{t}+1}{2}\alpha - 3\frac{\mathrm{t}+1}{2} \le \mathrm{t}(\alpha-1)+1$. Now assume $\{p_1, p_2, p_3\} \subset A$ and hence $t \ge 2$. We get $\mathrm{w_A(W')} \le \frac{\mathrm{t}+1}{2}\alpha - 2\mathrm{t} \le \mathrm{t}(\alpha-1)+1$.

Now assume $t = n-1$ and $A \neq H'$. In the case $\sharp(S \cap A) = n$ we conclude as in the case $t \le n-2$. Assume $\sharp(S \cap A) = n+1$. Obviously $\sharp(S \cap A \cap H') \ge n-1$, hence $\mathrm{w_A(W')} \le \mathrm{w_A(Z)} - \mathrm{n} + 1$. Moreover $S \cap (A \setminus H') \subseteq \{p_1, p_2, p_3\}$ hence $\mathrm{w_A(Z)} \le \mathrm{m_{n+3}} + \cdots + \mathrm{m_5} + \mathrm{m_3} + \mathrm{m_2}$. By Lemma 2.5 with $\eta = \alpha$, we have $\mathrm{w_A(Z)} \le \frac{\mathrm{n}+1}{2}\alpha - \frac{\mathrm{n}-1}{2}$. Therefore we conclude that $\mathrm{w_A(W')} \le \frac{\mathrm{n}+1}{2}\alpha - 3\frac{\mathrm{n}-1}{2} \le (\mathrm{n}-1)(\alpha-1)+1$.

□

3. The bound for the regularity index from [3]

Brambilla, Dumitrescu and Postinghel [3] gave a bound on the sum of the multiplicities for a linear system interpolating points in general position in $\mathbb{P}^n$ to be only linearly obstructed.

The notion of *general position* adopted in this paper is given by the following condition. Let $(\mathbb{P}^n)^{[s]}$ be the Hilbert scheme parametrizing s points of $\mathbb{P}^n$, and let $\mathcal{S}$ denote the point in $(\mathbb{P}^n)^{[s]}$ corresponding to a set S of s distinct points in $\mathbb{P}^n$. The set $S \subset \mathbb{P}^n$ is in general position if $\mathcal{S}$ belongs to a Zariski open subset of $(\mathbb{P}^n)^{[s]}$.

In particular, a set of points $S \subset \mathbb{P}^n$ in general position is in linearly general position, it does not contain more than $n+3$ points that lie on a rational normal curve of degree n, etc.

Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ be the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ passing through a collection of s points in general position with multiplicities at least $m_1, \ldots, m_s$.

Definition 3.1. The *(affine) virtual dimension* of $\mathcal{L}$ is defined by

$$\mathrm{vdim}(\mathcal{L}) = \binom{n+d}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n}$$

and the *expected dimension* of $\mathcal{L}$ is $\mathrm{edim}(\mathcal{L}) = \max(\mathrm{vdim}(\mathcal{L}), 0)$. If $\dim(\mathcal{L}) = \mathrm{edim}(\mathcal{L})$, or equivalently then $\mathcal{L}$ is said to be *non-special*.

Remark 3.2. Notice that $\mathrm{vdim}(\mathcal{L}) = h^0(\mathbb{P}^n, \mathcal{L}) - h^1(\mathbb{P}^n, \mathcal{L})$, hence $\mathcal{L}$ is non-special if and only if $h^1(\mathbb{P}^n, \mathcal{L}) = 0$.

Definition 3.3 ([3, Definition 3.2])**.** For any integer $-1 \le r \le s-1$ and for any multi-index $I(r) = \{i_1, \ldots, i_{r+1}\} \subseteq \{1, \ldots, s\}$, define the integer

$$k_{I(r)} := \max(m_{i_1} + \cdots + m_{i_{r+1}} - rd, 0). \tag{6}$$

The *(affine) linear virtual dimension* of $\mathcal{L}$ is the number

$$\sum_{r=-1}^{s-1} \sum_{I(r) \subseteq \{1,\ldots,s\}} (-1)^{r+1} \binom{n + k_{I(r)} - r - 1}{n}. \tag{7}$$

where we set $I(-1) = \emptyset$. The *(affine) linear expected dimension* of $\mathcal{L}$, denoted by $\mathrm{ldim}(\mathcal{L})$, is defined as follows: it is 0 if $\mathcal{L}$ is contained in a linear system whose linear virtual dimension is non-positive, otherwise it is the maximum between the linear virtual dimension of $\mathcal{L}$ and 0. If $\dim(\mathcal{L}) = \mathrm{ldim}(\mathcal{L})$, then $\mathcal{L}$ is said to be *only linearly obstructed*.

Asking whether the dimension of a given linear system equals its linear expected dimension can be thought as a refinement of the classical question of asking whether the dimension equals the expected dimension.

Theorem 3.4. [3, Theorem 5.3] *Set $s \geq n+3$, $n \geq 1$, $d \geq 2$ and $d \geq m_1 \geq \cdots \geq m_s \geq 1$. Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ be a linear system with points in general position. Let $s(d)$ be the number of multiplicities equal to d, namely the smallest integer such that $m_{s(d)+1} < d$. Assume that*

$$\sum_{i=1}^{s} m_i \leq nd + b, \tag{8}$$

where $b = b(\mathcal{L}) := \min\{n - s(d), s - n - 2\}$. Then $\mathcal{L}$ is only linearly obstructed, i.e. $\dim(\mathcal{L}) = \operatorname{ldim}(\mathcal{L})$.

As an easy consequence of the above result, one obtains the following.

Corollary 3.5. *If the same hypotheses as in Theorem 3.4 are satisfied and moreover*

$$d \geq m_1 + m_2 - 1, \tag{9}$$

then $\mathcal{L}$ is non-special, namely $\dim(\mathcal{L}) = \operatorname{edim}(\mathcal{L})$.

Proof. By Theorem 3.4, $\mathcal{L}$ is only linear obstructed. In fact $\dim(\mathcal{L}) = \operatorname{ldim}(\mathcal{L}) = \operatorname{vdim}(\mathcal{L})$, hence $h^1(\mathcal{L}) = 0$ by Remark 3.2. Indeed (9) implies that the line spanned by the first two points is contained at most simply in the base locus of $\mathcal{L}$ hence it does not create speciality. Because $m_1 \geq \cdots \geq m_s$, the same is true for all other lines and for all higher dimensional cycles spanned by subsets of Z_{red}. □

We can rephrase the above results and give an upper bound for the regularity index of a collection of fat points in general position in $\mathbb{P}^n$. For a linear system $\mathcal{L}$, let us define the positive integer

$$c = c(\mathcal{L}) := \min\{n, s - n - 2\}. \tag{10}$$

Theorem 3.6. *Let Z be a collection of fat points in general position in $\mathbb{P}^n$ with multiplicities $m_1 \geq \cdots \geq m_s \geq 1$. Then*

$$\mathrm{reg}(\mathrm{Z}) \leq \max\left\{\mathrm{m}_1 + \mathrm{m}_2 - 1, \left\lceil \frac{\mathrm{w(Z)} - \mathrm{c}}{\mathrm{n}} \right\rceil \right\}. \tag{11}$$

Proof. If d is bigger or equals the number on the right hand side of (11), then the linear system $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$ is non-special, by Corollary 3.5. □

Remark 3.7. When $m_1 + m_2 - 1 \geq \lceil (\mathrm{w(Z)} - \mathrm{c})/\mathrm{n} \rceil$ the bound in Theorem 3.6 is sharp. Indeed if $d < m_1 + m_2 - 1$, then the corresponding linear system $\mathcal{L}$ has $h^1(\mathcal{L}) > 0$, because it contains the line spanned by the first two points with multiplicity at least two in its base locus. This implies that $\mathrm{reg(Z)} \geq \mathrm{m}_1 + \mathrm{m}_2 - 1$.

In the rest of the section, we will make a comparison between the bounds (3) and (11) in the case of points in general position.

3.1. **Case** $s = n+3$**.** One can easily check that if $s = n+3$, the bound (11) coincides with Segre's bound (3). To see this, let us denote by μ, λ the integers such that $\mathrm{w(Z)} = \mu\mathrm{n} + \lambda$, with $0 \leq \lambda \leq n-1$. Since in this case $c = 1$ (with c defined as in (10)), one can easily check that $\lceil(\mathrm{w(Z)} - \mathrm{c})/\mathrm{n}\rceil = \lfloor(\mathrm{w(Z)} + \mathrm{n} - 2)/\mathrm{n}\rfloor$ equals μ when $\lambda \leq 1$ and it equals $\mu + 1$ when $\lambda \geq 2$.

Since $s = n+3$ always lie on a rational normal curve of degree n in $\mathbb{P}^n$, Proposition 3.6 provides a different proof of ([9, Proposition 7]) in this case.

3.2. **Quasi-homogeneous case.** For a quasi-homogeneous scheme Z, containing $s-1$ simple points and one fat point of weight d in general position, it is easy to see that its regularity index, reg(Z), equals d as long as $s \leq \binom{n-1+d}{d}$, obviously improving the bound (11). Indeed, it follows easily that a linear system of the form $\mathcal{L}_{n,d}(d, 1^{s-1})$ has the same dimension as the linear system $\mathcal{L}_{n-1,d}(1^{s-1})$. The bound obtained in Theorem 3.6 for the regularity index is d. This improves the bound (3) as soon as $s \geq nd - d + 3$. We leave it to the reader to check the details.

3.3. **Case** $s \geq n+4$**.** Table 1 and Table 2 contain the comparison between the second terms of the bounds (3) and (11).

	$\lambda = 0$, $s = 2n+2$	$\lambda = 0$, $s \leq 2n+1$	$\lambda = 1$	$2 \leq \lambda \leq s-n-2$	$s-n-2 < \lambda \leq n-1$
(3)	μ	μ	μ	$\mu+1$	$\mu+1$
(11)	$\mu-1$	μ	μ	μ	$\mu+1$

TABLE 1. Comparison table in the case $n+4 \leq s \leq 2n+2$.

	$\lambda = 0$	$\lambda = 1$	$2 \leq \lambda \leq n-1$
(3)	μ	μ	$\mu+1$
(11)	$\mu-1$	μ	μ

TABLE 2. Comparison table in the case $s \geq 2n+3$.

We conclude this section with a list of examples in which Theorem 3.6 provides an improvement of the Segre's bound. The bounds we present in the examples below are sharp, in other words, the regularity index for the corresponding scheme of fat points is given by (11).

Example 3.8. For the planar case $n = 2$, the bound (11) improves the bound given by Segre (1). One can easily check this by considering the scheme given by six double points for which Segre's bound (1) equals 6. However the linear system of quintic curves $\mathcal{L} = \mathcal{L}_{2,5}(2^6)$ has vanishing cohomology group $H^1(\mathbb{P}^2, \mathcal{L})$, as predicted by our bound (11) that is 5.

Example 3.9. For a scheme of nine double points in $\mathbb{P}^3$, Segre's bound (3) is 6, but $\mathcal{L} = \mathcal{L}_{3,5}(2^9)$ has vanishing $H^1(\mathbb{P}^3, \mathcal{L})$, as predicted by the bound (11) that is 5.

4. Modification of Segre's conjecture for arbitrary number of points

In this section we introduce the following conjecture for points in arbitrary position and present the evidences we have for it.

Conjecture 4.1. *Fix integers $d \geq 2$, $n > 2$. Let $S \subset \mathbb{P}^n$ be a finite collection of s points. Fix integers $m_1, \dots, m_s$ and set $Z := \sum_{i=1}^{s} m_i p_i$. Then $h^1(\mathcal{I}_Z(d)) = 0$ if all the following conditions are satisfied:*

(1) $\mathrm{w(Z) \leq nd + 1}$*;*
(2) $\mathrm{w_L(Z) \leq d + 1}$ *for each line $L \subset \mathbb{P}^n$;*
(3) for all integers $r = 2, \dots, n-1$ and every r-dimensional linear subspace $L \subset \mathbb{P}^n$ we have $\mathrm{w_L(Z) \leq rd + 2}$*; if* $\mathrm{w_L(Z) = rd + 2}$ *we also assume $h^1(L, \mathcal{I}_{Z \cap L}(d)) = 0$.*

Condition (1) (resp. (2)) of Conjecture 4.1 is to make sure that no rational normal curve (resp. line), spanned by points of S, is contained in the singular locus of $\mathcal{I}_Z(d)$. See also Remark 4.3 below. Moreover condition (3) says that if neither does any higher dimensional linear spaces L spanned by points of S (namely $\mathrm{w_L(Z) \leq d + 1}$), then $h^1(\mathcal{I}_Z(d)) = 0$. When $\mathrm{w_L(Z) = d + 2}$, condition (3) says that if both (1) and (2) are satisfied, the non-vanishing $h^1(\mathcal{I}_Z(d)) > 0$ happens because of the non-vanishing of the first cohomology of the same sheaf restricted to L.

Lemma 4.2. *Assume the existence of a closed subscheme $W \subsetneq Z$ such that $h^1(\mathcal{I}_W(d)) > 0$. Then $h^1(\mathcal{I}_Z(d)) > 0$.*

Proof. Since $\dim(Z) = 0$, we have $h^1(Z, \mathcal{I}_{W,Z}(d)) = 0$. Hence the restriction map $H^0(\mathcal{O}_Z(d)) \to H^0(\mathcal{O}_W(d))$ is surjective. Since $h^1(\mathcal{I}_W(d)) > 0$, we get $h^1(\mathcal{I}_Z(d)) > 0$. □

Remark 4.3. In the same notation as in Conjecture 4.1, assume the existence of a closed subscheme $T \subset \mathbb{P}^n$ such that $h^0(T, \mathcal{O}_T(d)) < \mathrm{w_T(Z)}$. Then $h^1(T, \mathcal{I}_{Z \cap T}(d)) > 0$. Hence $h^1(\mathcal{I}_{Z \cap T}(d)) > 0$. Lemma 4.2 gives $h^1(\mathcal{I}_Z(d)) > 0$. In particular to have any chance that $h^1(\mathcal{I}_Z(d)) = 0$, we need $\mathrm{w_C(Z) \leq md + 1}$ for every rational normal curve of degree m of an m-dimensional linear subspace of $\mathbb{P}^n$ (we allow the case $m = n$). In particular condition (2) of Conjecture 4.1 is a necessary condition.

Theorem 4.4. *Fix integers $n \geq 2$, $d \geq 4$, $d \geq m_1 + 2$, $s \geq 2n+3$, $m_1 \geq \cdots \geq m_s > 0$, $\sum_i m_i = nd + 2$, $m_1 + m_{2n+2} \leq d$ and $m_1 + m_2 \leq d + 1$. Let $S = \{p_1, \dots, p_s\} \subset \mathbb{P}^n$ be a finite subset of points in linearly general position. Set $Z := \sum_i m_i p_i$. We have $h^1(\mathcal{I}_Z(d)) = 0$ if and only if S is not contained in a rational normal curve of degree n of $\mathbb{P}^n$.*

Proof. If S is contained in a rational normal curve of $\mathbb{P}^n$, then $h^1(\mathcal{I}_Z(d)) > 0$, see [9, Proposition 7]. We use induction on d starting from the case $d = \max\{4, m_1 + 2\}$. We check in cases (d), (e) and (f) the starting cases of the induction. In cases (a) and (b) we do not use the inductive assumption, but reduce the proof to a game with a scheme W with $h^1(\mathcal{I}_W(d-2)) = 0$ by [9, Theorem 6].

Assume $h^1(\mathcal{I}_Z(d)) > 0$. For each $p \in S$ let m_p be the multiplicity of p in Z. Let B_1 (resp. B_1') be the set of all lines $L \subset \mathbb{P}^n$ such that $\mathrm{w_L(Z) = d + 1}$ (resp. $\mathrm{w_L(Z) = d}$). Since S is in linearly general position, $L \in B_1$ (resp. B_1') if and only if L is a line spanned by two different points, say p and q, with $m_p + m_q = d + 1$ (resp. $m_p + m_q = d$). In particular if $B_1 \neq \emptyset$, then $m_1 + m_2 = d + 1$. If $m_1 > m_2$, then

the elements of B_1 are the lines spanned by p_1 and one the points p_i, $2 \le i \le a$, where a is the maximal integer $i \in \{2, \dots, s\}$ with $m_i = m_2$. If $m_1 = m_2$, then either $B_1 = \emptyset$ (case d even) or $B_1' = \emptyset$ (case d odd) and $B_1 \cup B_1' \ne \emptyset$ if and only if $m_1 = m_2 = \lceil d/2 \rceil$. In this case the elements of $B_1 \cup B_1'$ are the lines spanned by two points of $\{1, \dots, a'\}$, where a' is the maximal integer $i \le s$ with $m_i = m_1 = m_2$. We consider the following cases.

Case (a): $m_1 = m_2$ and d odd.

Let $Q \subset \mathbb{P}^n$ be a general quadric hypersurface containing $\{p_1, \dots, p_{2n+1}\}$. Such a quadric exists, because $2n+1 < \binom{n+2}{2}$. Set $W := \mathrm{Res}_Q(Z)$ and $S' := W_{\mathrm{red}}$. For each $p \in S$ let u_p be the multiplicity of p in W. If $p = p_i$, set $u_i := u_{p_i}$. We have $u_p = m_p$ if $p \notin Q$, $u_p = m_p - 1$ if p is a smooth point of Q and $u_p = \max\{0, m_p - 2\}$ if p is a singular point of Q. Since $S' \subseteq S$, S' is in linearly general position. We assume $\sharp(S') \ge n+1$, i.e. that S' spans $\mathbb{P}^n$; see the proof of case (e) for the case $\sharp(S') \le n$. Since $\sharp(S \cap Q) \ge 2n+1$, we have $\sum_{p \in S'} u_p \le nd+2-2n-1 \le n(d-2)+1$. If $B_1 \cup B_1' = \emptyset$, then $\mathrm{w_L(W)} \le \mathrm{d} - 1$ for all lines $L \subset \mathbb{P}^n$ and hence $h^1(\mathcal{I}_W(d-2)) = 0$ ([9, Theorem 6]). Now assume $B_1 \cup B_1' \ne \emptyset$ and let e be the maximal integer $i \le s$ with $m_i = \lceil d/2 \rceil$. Since $\sum_i m_i = nd + 2 < (2n+1)\lceil d/2 \rceil$, we have $e \le 2n$. Hence if $L \in B_1 \cup B_1'$, then $\mathrm{w_L(W)} \le \mathrm{w_L(Z)} - 2 \le \mathrm{d} - 1$ and hence $h^1(\mathcal{I}_W(d-2)) = 0$ ([9, Theorem 6]). Since $h^1(\mathcal{I}_Z(d)) > 0$, the Castelnuovo residual sequence with respect to Q gives $h^1(Q, \mathcal{I}_{Z \cap Q}(d)) > 0$. Since the zero-dimensional scheme $Z \cap Q$ is contained in the scheme $Z' := \sum_{p \in S \cap Q} m_p p$, we get $h^1(\mathcal{I}_{Z'}(d)) > 0$ (Lemma 4.2). The support of Z' is in linearly general position and $\mathrm{w(Z' \cap L)} \le \mathrm{w(Z \cap L)} \le \mathrm{d} + 1$ for all lines L. Hence by [9, Theorem 6] we have $\sum_{p \in S \cap Q} m_p \ge nd + 2$, i.e. $S \cap Q = S$, i.e. S is in the base locus of $|\mathcal{I}_{\{p_1, \dots, p_{2n+1}\}}(2)|$, i.e. $|\mathcal{I}_{\{p_1, \dots, p_{2n+1}\}}(2)| = |\mathcal{I}_S(2)|$ and $h^0(\mathcal{I}_S(2)) = \binom{n+2}{2} - 2n - 1$. Since S is in linearly general position, $h^1(\mathcal{I}_A(2)) = 0$ for all $A \subset S$ with $\sharp(A) \le 2n+1$. By [17, Lemma 3.9], S is contained in a rational normal curve.

Case (b): $m_1 = m_2$ and d even.

Notice that in this case $B_1 = \emptyset$. If $B_1' = \emptyset$, i.e. if $m_1 < d/2$, then we make the same construction as in Case (a) with $e = 0$. We still have $\mathrm{w_L(W)} \le \mathrm{d} - 1$ for all lines $L \subset \mathbb{P}^n$, because $\mathrm{w_L(Z)} \le \mathrm{d} - 1$ for every line L and so $h^1(\mathcal{I}_W(d-2)) = 0$. Now assume $m_1 = d/2$. Let f be the maximal integer $i \le s$ such that $m_i = d/2$. Since $s \ge 2n+3$ and $1 + (2n+2)d/2 > nd+2$, we have $f \le 2n+1$. We may repeat the proof of Case (a), because $\mathrm{w_L(W)} \le \mathrm{d} - 1$ holds.

Case (c): $m_1 > m_2$ and $d > 4$.

If $m_1 + m_2 \le d-1$ and $d \ge 6$, then we may repeat the proof of Case (a), because $\mathrm{w_L(W)} \le \mathrm{d} - 1$ for all lines L (see the details in case (e)). Hence we may assume $m_1 + m_2 \ge d$. Let x be the maximal integer $i \le s$ such that $m_i = m_2 = d + 1 - m_1$ (with the convention $x = 0$ if there is no such an integer, i.e. if $m_2 = d - m_1$) and let g be the maximal integer $i \le s$ such that $m_i = d - m_1$ with the convention $g = x$ if there is no such an integer. If $g \le 2n+1$, then we may repeat the proof of Case (a), because $\mathrm{w_L(W)} \le \mathrm{d} - 1$ for all lines L. Hence we may assume $g \ge 2n+2$. Let H be the hyperplane spanned by $p_1, \dots, p_n$. Set $U := \mathrm{Res}_H(Z)$ and let $S_1 := U_{\mathrm{red}}$ be the support of U. For each $p_i \in S$ let r_i be the multiplicity of p_i in U. We have $r_i = m_i - 1$ if $i \le n$ and $r_i = m_i$ if $i > n$. Since each element of B_1 contains p_1, we have $\mathrm{w_L(U)} \le \mathrm{d}$ for all lines L. We have $\sum_i r_i = n(d-1) + 2$. Since $d \ge m_1 + 2$ and $g \ge 2n+2$, then $m_i \ge 2$ for all $i \le g$, in particular for all $i \le n$. Hence

$S_1 = S$ and $\sharp(S_1) \geq 2n+3$. We order the sequence $r_1, \dots, r_s$ in non-decreasing order $y_1 \geq \cdots \geq y_s$. We have $y_1 = r_1 = m_1 - 1$. Since $d \geq m_1 + 2$, $m_1 > m_2$, $r_1 = m_1 - 1$ and $y_i \leq m_i$ for all i, we have $d-1 \geq y_1 + 2$. We have $y_1 + y_{2n+2} \leq d-1$, because $y_{2n+2} \leq m_{2n+2}$ and $y_1 = m_1 - 1$. Hence by the inductive assumption there is a rational normal curve $C \supset S_1 = S$.

Case (d): Assume $d = 4$ and so $m_1 \leq 2$. Let f be the maximal integer i such that $m_i = 2$ with the convention $f = 0$ if $m_1 = 1$. Since $\mathrm{w(Z)} = 4\mathrm{n} + 2$, we have $f \leq 2n+1$. Let Q be a general quadric hypersurface containing $\{p_1, \dots, p_{2n+1}\}$. Set $W := \mathrm{Res}_Q(Z)$. For each $p \in S$ set $m'_p = m_p$ if $p \notin Q$ and $m'_p = m_p - 1$ if $p \in Q$. Set $Z' = \sum_{p \in S} m'_p p$. Since $f \leq 2n+1$, W is a reduced scheme with cardinality $\leq 2n+1$ and in linearly general position. Therefore $h^1(\mathcal{I}_W(2)) = 0$. The Castelnuovo's sequence of Q gives $h^1(Q, \mathcal{I}_{Z\cap Q}(4)) > 0$. Since $Z' \cap Q = Z \cap Q$, Lemma 4.2 gives $h^1(\mathcal{I}_{Z'}(4)) > 0$. Since $Z'_{\mathrm{red}} \subseteq S$ is in linearly general position, [9, Theorem 6] gives $\mathrm{w(Z')} > 4\mathrm{n} + 1$, i.e. $Z' = Z$. Hence $Q \supset S$. Since a general quadric hypersurface containing $\{p_1, \dots, p_{2n+1}\}$ contains S, S is in linearly general position and $\sharp(S) \geq 2n+3$, then S is contained in a rational normal curve ([17, Lemma 3.9]).

Case (e) Assume $d = 5$ and hence $m_1 \leq 3$. Let h be the maximal integer i such that $m_i \geq 3$. Since $\mathrm{w(Z)} = 5\mathrm{d} + 2$, we have $h \leq 2n+1$. Take Q and $W = \mathrm{Res}_Q(Z)$ as in Case (e). We have $\mathrm{w(W)} \leq 3\mathrm{n} + 1$ and W_{red} is in linearly general position. Call $u_1 \geq u_2 \geq \cdots$ the multiplicities in W of the points of W_{red}. Since $h \leq 2n+1$, we $u_1 \leq 2$ and hence $u_1 + u_2 \leq 4$. If W_{red} spans $\mathbb{P}^n$, then [9, Theorem 6] gives $h^1(\mathcal{I}_W(3)) = 0$ and we conclude as in Case (d). Now assume that W_{red} spans a linear subspace M of dimension $r < n$. Write $W_{\mathrm{red}} = \{q_1, \dots, q_{r+1}\}$ with q_i appearing with multiplicity u_i in W. By [9, Theorem 6] we have $h^1(M, \mathcal{I}_{W\cap M}(3)) = 0$. Take a linear subspace $M \subset N \subseteq \mathbb{P}^n$ with $\dim(N) = r+1$. See M as a hyperplane of N to compute the residual scheme $\mathrm{Res}_M(W \cap N)$ of the scheme $W \cap N \subset N$. Each q_i occurs in $\mathrm{Res}_N(W \cap N)$ with multiplicity $u_i - 1 \leq 1$. Hence $h^1(N, \mathcal{I}_{\mathrm{Res}_M(N\cap Z)}(1)) = 0$ and so $h^1(N, \mathcal{I}_{\mathrm{Res}_M(N\cap Z)}(2)) = 0$. The Castelnuovo's sequence of M gives $h^1(N, \mathcal{I}_{N\cap W}(3)) = 0$. If $N = \mathbb{P}^n$, then $h^1(\mathcal{I}_W(3)) = 0$. If $r+1 < n$ we take a flag of linear subspaces $N \subset N_1 \subset \cdots \subset N_{n-r-1} = \mathbb{P}^n$ with $\dim(N_i) = r+1-i$ for all i. After $n-r-1$ steps we get $h^1(\mathcal{I}_W(3)) = 0$. Then we conclude as in Case (d)

Case (f) Assume $d = m_1 + 2 \geq 6$. Since $m_1 + m_2 \leq d+1$, we have $m_2 \leq 3$. Let g be the maximal integer i such that $m_g \geq 3$. Since $m_{2n+2} + m_1 \leq d$, we have $g \leq 2n+1$. Take Q and W as in Case (e). As in Case (a) or (d) it is sufficient to prove that $h^1(\mathcal{I}_W(d-2)) = 0$. Call $u_1 \geq u_2 \geq \cdots \geq$ the multiplicities in W of the points of W_{red}. Since $g \leq 2n+1$, we have $u_2 \leq 2$. Since $u_1 \leq m_1 - 1 \leq d-3$ and W_{red} is in linearly general position, we have $h^1(\mathcal{I}_W(d-2)) = 0$ and we conclude as in the last two cases. $\square$

4.1. **Conjecture 4.1 holds for $n = 3$.** In this section we prove that Conjecture 4.1 holds for $n = 3$.

Proposition 4.5. *Conjecture 4.1 is true if $n = 3$.*

Proof. Let d be an integer and Z a fat point scheme for which conditions (1), (2) and (3) of Conjecture 4.1 are satisfied, but $h^1(\mathcal{I}_Z(d)) > 0$. Set $S := Z_{\mathrm{red}}$. By [13] or [22] Segre's conjecture holds in $\mathbb{P}^3$. Hence we may assume the existence of a plane

$H \subset \mathbb{P}^3$ such that $\mathrm{w_H(Z)} = 2\mathrm{d}+2$. By assumption we have $h^1(H, \mathcal{I}_{Z\cap H,H}(d)) = 0$. Hence the residual sequence of H and Z gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$. Since $\mathrm{w_L(Z)} \le \mathrm{d}+1$ for every line and $\mathrm{w_H(Z)} = 2\mathrm{d}+2$, we have $\sharp(H \cap S) \ge 4$. Hence $\mathrm{w(Res_H(Z))} \le 3(\mathrm{d}-1)+1$. Fix a line $L \subset \mathbb{P}^3$. Since $\mathrm{w_L(Z)} \le \mathrm{d}+1$, we have $\mathrm{w_L(Res_H(Z))} \le \mathrm{d}+1$ with equality if and only if $\mathrm{w_L(Z)} = \mathrm{d}+1$ and $S\cap L\cap H = \emptyset$. If $\mathrm{w_L(Z)} = \mathrm{d}+1$ and $S \cap L \cap H = \emptyset$, then $\mathrm{w(Z)} \ge \mathrm{w_H(Z)} + \mathrm{w_L(Z)} = 3\mathrm{d}+3$, a contradiction. Therefore $\mathrm{Res}_H(Z)$ satisfies the Segre condition (Conjecture 1.1) with respect to lines, i.e. $\mathrm{w_L(Res_H(Z))} \le \mathrm{d}$ for all lines $L \subset \mathbb{P}^3$. Since Segre condition is true even in degree 1 by [13] or [22], the fact that $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$ implies the existence of a plane $M \subset \mathbb{P}^2$ such that $\mathrm{w_M(Res_H(Z))} \ge 2(\mathrm{d}-1)+1 = 2\mathrm{d}-1$.

If $\sharp(S\cap H\cap M) \ge 4$, then we get $\mathrm{w_M(Z)} \ge \mathrm{w_M(Res_H(Z))} + 4 \ge 2\mathrm{d}+3$, which is in contradiction with assumption (3). Therefore $\sharp(S \cap H \cap M) \le 3$. Notice that in this case, since $\sharp(S\cap H) \ge 4$, we have $M \ne H$. Since $M \cap H$ is a line, we have $\mathrm{w_{H\cap M}(Res_H(Z))} \le \mathrm{d}$. We have $3d+1 \ge \mathrm{w(Z)} \ge \mathrm{w_H(Z)} + \mathrm{w_M(Res_H(Z))} - \mathrm{w_{H\cap M}(Res_H(Z))} \ge 2\mathrm{d}+2+2\mathrm{d}-\mathrm{d}$. This gives a contradiction.

□

For a specific fat point scheme Z, a good starting point in order to understand the value of $h^1(\mathcal{I}_Z(d))$ is to consider linear subspaces V such that $h^1(V, \mathcal{I}_{Z\cap V,V}(d)) = 0$ and $\mathrm{w_V(Z)} \gg \mathrm{dim(V)}\cdot \mathrm{d}$. Then one should look at the residual exact sequences with respect to hyperplanes or hyperquadrics T containing V and for which $\mathrm{w_T(Z)}$ is large.

4.2. **Achieving Segre's bound.** A finite subset $A \subset \mathbb{P}^n$ is said to be in *uniform position* or that it has the *uniform position property* if any two subsets of A with the the same cardinality have the same Hilbert function. We recall that [9, Theorem 6] proves the Segre's bound (3) for points in linearly general position

$$\mathrm{reg(Z)} \le \max\left\{\mathrm{m_1} + \mathrm{m_2} - 1, \left\lfloor \frac{\mathrm{w(Z)} + \mathrm{n} - 2}{\mathrm{n}} \right\rfloor \right\}.$$

Moreover, [9, Proposition 7] shows that equality holds if the set $\{p_1, \dots, p_s\}$ is contained in a rational normal curve of $\mathbb{P}^n$. In [9, Problem 1], the authors asked if the equality in (3) implies that $\{p_1, \dots, p_s\}$ is contained in a rational normal curve of $\mathbb{P}^n$. It was shown to be true in [5] and [23, Theorem 2.1] for points in uniform position (at least if $m_{2n+3} \ge n$). We will show that this is not always the case by exhibiting the following family of examples.

Example 4.6. Fix positive integers $n \ge 4$, $m_i > 0$ and $d \ge 4$ such that $m_1 + \cdots + m_s = nd + \alpha$ with $1 \le \alpha \le n-1$. Assume the existence of an integer $b \in \{s-\alpha, \dots, s-1\}$ such that $\sum_{i=b+1}^{s} m_i \le \alpha$; for instance, take $b = n-\alpha$ and $m_i = 1$ for all $i > b$. Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree n. Take $p_i \in C$, $1 \le i \le b$, distinct and $p_j \in \mathbb{P}^n \setminus C$, $b+1 \le j \le s$, with the only restriction that the set $\{p_1, \dots, p_s\}$ is in linearly general position (e.g. we take p_j general for $j > b$). Set $Z' := \sum_{i=1}^{b} m_i p_i$. By [9, Proposition 7], we have $\mathrm{reg(Z')} = \mathrm{t}$, where t is the Segre's bound. By [9, Proposition 5], we have $\mathrm{reg(Z)} \le \mathrm{t}$. Lemma 4.2 implies that $\mathrm{reg(Z')} \le \mathrm{reg(Z)}$. Hence $\mathrm{reg(Z)} = \mathrm{t}$. If $s - b \ge n+3$, then C is the only rational normal curve containing $p_1, \dots, p_b$. Hence $\{p_1, \dots, p_s\}$ is contained in no rational normal curve.

Fix any positive integer x. We say that A is in *uniform position in degree* $\le x$ if for any $E, F \subset A$ with $\sharp(E) = \sharp(F)$ we have $h_E(t) = h_F(t)$ for $t = 1, \dots, x$. Uniform position in degree ≤ 1 is equivalent to linearly general position.

The proof of [23, Theorem 2.1] works verbatim just assuming that the set has uniform position in degree ≤ 2. Trung and Valla gave another result in which their conjecture is true and with the set only assumed to be in linearly general position ([23, Theorem 1.6]). So we do not see a natural way to improve Theorem 4.4 to the case $\mathrm{w(Z)} > \mathrm{nd} + 2$, nor to weaken the assumption $m_1 + m_{2n+2} \le d$.

In the following example the set S is in uniform position.

Example 4.7. Fix integers $d \ge 2$ and $n \ge 3$ and set $s := (n-1)d + 3$. Fix a point $p \in \mathbb{P}^n$, a hyperplane $H \subset \mathbb{P}^n$ such that $p \notin H$ and a rational normal curve $C \subset H$ of degree $n-1$. Let $T \subset \mathbb{P}^n$ be the cone with vertex p over C. Fix a general $S' \subset T$ with $\sharp(S') = s - 1$ and set $S := \{p\} \cup S'$. Set $m_p := d$ and $m_q = 1$ for all $q \in S'$. Write $Z := \sum_{q \in S} m_q q$. Let $\ell : \mathbb{P}^n \backslash \{p\} \to H$ denote the linear projection from p onto H. Set $A := \ell(S')$. We have $\sharp(A) = (n-1)d + 2$. Since A is contained in a rational normal curve of H, we have $h^1(H, \mathcal{I}_A(d)) = 1$. The linear system $|\mathcal{I}_{dp}(d)|$ is the set of all degree d cones with vertex containing p. Hence $h^0(\mathcal{I}_Z(d)) = h^0(H, \mathcal{I}_A(d))$ and $h^1(\mathcal{I}_Z(d)) = 1$. Since $s \ge n+4$ and S' is general in T, S is not contained in a rational normal curve of degree n of $\mathbb{P}^n$. We claim that S is in uniform position. To see this, fix an integer $k \ge 0$. Since T is an irreducible variety and $\mathcal{O}_{\mathbb{P}^n}(k)$ has no base points, for a general finite set $E \subset T$ we have $h^0(\mathcal{I}_E(k)) = \max\{h^0(\mathcal{I}_T(k)), \binom{n+k}{n} - \sharp(E)\}$ and $h^0(\mathcal{I}_{E \cup \{p\}}(k)) = \max\{h^0(\mathcal{I}_T(k)), \binom{n+k}{n} - \sharp(E) - 1\}$. Since S' is general in T, any two subsets of S with the same cardinality have the same Hilbert function.

Appendix A. Macaulay 2 Codes

In this section we provide Macaulay 2 [18] scripts for the computations that cover the initial cases of the proofs of the results contained in Section 2.

A.1. Code 1.

```
KK=ZZ/32749;
R=KK[e_0..e_4]

d=5;
N=binomial(d+4,4);

f=ideal(e_0..e_4);
fd=f^d;
T=gens gb(fd)
J=jacobian(T);
JJ=jacobian(J);

p0=matrix{{e_0^0,0,0,0,0}};
p1=matrix{{0,e_0^0,0,0,0}};
p2=matrix{{0,0,e_0^0,0,0}};
p3=matrix{{0,0,0,e_0^0,0}};
p4=matrix{{e_0^0,e_0^0,e_0^0,e_0^0,0}};
p5=matrix{{0,0,0,0,e_0^0}};
p6=matrix{{e_0^0,e_0^0,-e_0^0,0,e_0^0}};
```

```
mat=random(R^1,R^N)*0;
mat=(mat||sub(JJ,p0));
mat=(mat||sub(JJ,p1));
mat=(mat||sub(JJ,p2));
mat=(mat||sub(JJ,p3));
mat=(mat||sub(JJ,p4));
mat=(mat||sub(JJ,p5));
mat=(mat||sub(JJ,p6));

r=rank mat;
exprank=7*binomial(6,4);

print(r,exprank)
```

A.2. **Code 2.**

```
KK=ZZ/32749;
R=KK[e_0..e_5]

d=3;
N=binomial(d+5,5);

f=ideal(e_0..e_5);
fd=f^d;
T=gens gb(fd)
J=jacobian(T);

p0=matrix{{e_0^0,0,0,0,0,0}};
p1=matrix{{0,e_0^0,0,0,0,0}};
p2=matrix{{0,0,e_0^0,0,0,0}};
p3=matrix{{0,0,0,e_0^0,0,0}};
p4=matrix{{0,0,0,0,e_0^0,0}};
p5=matrix{{0,0,0,0,0,e_0^0}};
p6=matrix{{e_0^0,e_0^0,e_0^0,e_0^0,0,0}};
p7=matrix{{e_0^0,e_0^0,0,0,e_0^0,e_0^0}};

mat=random(R^1,R^N)*0;
mat=(mat||sub(J,p0));
mat=(mat||sub(J,p1));
mat=(mat||sub(J,p2));
mat=(mat||sub(J,p3));
mat=(mat||sub(J,p4));
mat=(mat||sub(J,p5));
mat=(mat||sub(J,p6));
mat=(mat||sub(J,p7));

r=rank mat;
exprank=8*6;
```

```
print(r,exprank)
```

References

[1] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, J. Algebraic Geom. **4(2)**, 201–222 (1995).

[2] B. Benedetti, G. Fatabbi and A. Lorenzini, Segre's bound and the case of $n+2$ fat points of $\mathbb{P}^n$, Comm. Algebra 40 (2012), 395–403.

[3] M. C. Brambilla, O. Dumitrescu and E. Postinghel, On a notion of speciality of linear systems in $\mathbb{P}^n$, Trans. Am. Math. Soc. 367 (2015), 5447–5473

[4] M.C. Brambilla and G. Ottaviani, On the Alexander-Hirschowitz theorem, J. Pure Appl. Algebra 212 (2008), no. 5, 1229–1251.

[5] M. V. Catalisano, Linear systems of plane curves through fixed fat points of $\mathbf{P}^2$, J. Algebra 142 (1991), no. 1, 81–100.

[6] M. V. Catalisano, Fat points on a conic, Comm. Algebra 19 (1991) 2153–2168.

[7] M. V. Catalisano, P. Ellia and A. Gimigliano, Fat points on rational normal curves, J. Algebra 216 (1999), 600–619

[8] M. V. Catalisano and A. Gimigliano, On the Hilbert function of fat points on a rational normal cubic, J. Algebra 183 (1996), 245-265.

[9] M. V. Catalisano, N. V. Trung and G. Valla, A sharp bound for the regularity index of fat points in general position, Proc. Amer. Math. Soc. 118 (1993), no. 3, 717–724.

[10] E. D. Davis and A. V. Geramita, The Hilbert function of a special class of 1-dimensional Cohen-Macaulay graded algebras, in The Curves Seminar at Queen's, Queen's Papers in Pure and Appl. Math., vol. 67 (1984), 1–29.

[11] G. Fatabbi, Regularity index of fat points in the projective plane, J. Algebra 170 (1994) 916-928.

[12] G. Fatabbi, On the resolution of ideals of fat points, J. Algebra 242 (2001), no. 1, 92–108.

[13] G. Fatabbi and A. Lorenzini, On a sharp bound for the regularity index of any set of fat points. J. Pure Appl. Algebra 161 (2001), 91–111.

[14] W. Fulton, Algebraic Curves, Math. Lect. Note Series, Benjamin, New York, 1969.

[15] A. Gimigliano, Regularity of linear systems of plane curves, J. Algebra 124 (1989) 447–460.

[16] A. Gimigliano, Our thin knowledge on fat points, in: The Curves Seminar at Queen's VI, Queen's Papers in Pure and Appl. Math., vol. 83, 1989, pp. 1B-50B.

[17] J. Harris, with the collaboration of D. Eisenbud, Curves in Projective Space, Les Presses de l'Université de Montréal, Montréal, 1982.

[18] D. Grayson, M. Stillman, Macaulay 2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.

[19] E. Postinghel, A new proof of the Alexander-Hirschowitz interpolation theorem, Ann. Mat. Pura Appl. (4) 191 (2012), no. 1, 77-94.

[20] B. Segre, Alcune questioni su insiemi finiti di punti in Geometria Algebrica, in Atti del Convegno Internazionale di Geometria Algebrica, Torino (1961), 15–33.

[21] P. V. Thiên On Segre bound for the regularity index of fat points in $\mathbb{P}^2$, Acta Math. Vietnam. 24 (1999), no. 1, 75–81.

[22] P. V. Thiên, Segre bound for the regularity index of fat points in $\mathbb{P}^3$, J. Pure Appl. Algebra 151 (2000), 197–214.

[23] N. V. Trung and G. Valla, Upper bounds for the regularity index of fat points, J. Algebra 176 (1995), no. 1, 182–209.

[24] N. C. Tu and T. M. Hung, On the regularity index of n+3 almost equimultiple fat points in $\mathbb{P}^n$, Kyushu J. Math. 67 (2013), 203–313.

E-mail address: `ballico@science.unitn.it`

Dept. of Mathematics, University of Trento, 38050 Povo (TN), Italy

E-mail address: `dumitrescu@math.uni-hannover.de`

Institut für Algebraische Geometrie GRK 1463, Welfengarten 1, 30167 Hannover, Germany

E-mail address: `elisa.postinghel@wis.kuleuven.be`

KU Leuven, Department of Mathematics, Celestijnenlaan 200B, 3001 Heverlee, Belgium